

**PRE-SERVICE TEACHERS' HANDLING OF LINEAR ALGEBRA IN A
PROBLEM-CENTRED APPROACH**

SALIMMA GEORGE

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SUPERVISOR

Professor C. Julie

CO-SUPERVISOR

A. I. Olivier

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DECLARATION

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

SUMMARY

The primary concern of the study is how pre-service teachers perform after they have been exposed to a section of a linear algebra course based on the problem-centred approach. The students were in their final (3rd) year of a teacher education course at a college of education which prepares them to teach mathematics at high school level. Sixty students, who formed the experimental group, were exposed to a linear algebra section, which was underpinned by the tenets of the problem-centred approach. The control group comprised of 60 students of similar mathematical background and they were taught the linear algebra section in the conventional way. The main study is preceded by an overview of the history of the teaching of linear algebra and this overview rendered that certain aspects of linear algebra were historically taught in context. Furthermore an analysis of current secondary school mathematics curricula indicated that there are components of linear algebra present in these syllabi.

To test whether there was any significant effect of the experimental course, both groups were subjected to the same linear algebra test items at the end of the experimental period. The null hypothesis tested was: there will be no significant difference between the achievement scores of the experimental and control groups. A simple statistical two-tailed test for the difference between two means was done. This test confirmed the rejection of the null hypothesis at the 0,01 level of significance. It is thus accepted that the superior achievement of the experimental group was due to the intervention – approaching aspects of linear algebra through the problem-centred approach.

To get an indication of the strategies the experimental group followed to solve linear algebra problems, an analysis was done of the written work of the students. This analysis showed that students applied an absolute calculation strategy to seek solutions to the problems.

The study had the following limitations:

1. The students were not representative of the pre-service secondary teachers in South Africa. Only students from the developing population group were involved.
2. The students were not randomly assigned to the experimental and control group. They were in their normal college classes.

Notwithstanding the above limitations it is recommended that:

1. The problem-centred approach, which support the ideals of outcomes-based education, be applied to a major part of the South African school and college of education mathematics syllabi.
2. Appropriate assessment procedures consonant with the problem-centered approach are installed.
3. Adequate support systems are put in place to support teacher transition from the conventional to the problem-centred approach.

OPSOMMING

Die primêre fokus van die studie is die effek van 'n lineêre algebra kursus, aangebied volgens die probleem-gesentreerde benadering, op kollege onderwysstudente. Die studente was in hulle finale (3de) jaar van 'n kursus aan 'n onderwyskollege wat hulle voorberei om wiskunde op hoërskoolvlak te onderrig. Die eksperimentele groep, bestaande uit 60 studente, het aspekte van lineêre algebra geleer, onderrig volgens die probleem-gesentreerde benadering. Die kontrolegroep, bestaande uit 60 studente met omtrent dieselfde wiskunde agtergrond, het dieselfde lineêre algebra geleer, onderrig volgens die konvensionele metode.

Die hoofstudie is voorafgegaan deur 'n oorsig van die geskiedenis van die onderrig van lineêre algebra, wat getoon het dat sekere aspekte van lineêre algebra histories in konteks onderrig is. 'n Ontleding van die huidige hoërskool wiskunde-kurrikulum toon dat dit komponente van lineêre algebra bevat.

Om die impak van die eksperimentele kursus te bepaal, het beide groepe aan die einde van die eksperimentele periode dieselfde lineêre algebra toetsitems voltooi. Die volgende nul-hipotese is getoets: Daar is geen beduidende verskil tussen die prestasies van die eksperimentele en die kontrole groepe nie. 'n Eenvoudige tweevlerk statistiese toets vir die verskil tussen twee gemiddeldes is gedoen. Die toets bevestig die verwerping van die nul-hipotese op die 0,01 vlak van beduidendheid. Dit word dus aanvaar dat die beter prestasie van die eksperimentele groep toegeskryf kan word aan die intervensie, naamlik die leer van lineêre algebra volgens die probleem-gesentreerde benadering.

Om 'n aanduiding te kry van die strategieë wat die eksperimentele groep gebruik het in die oplos van lineêre algebra probleme, is die geskrewe werk van die studente ontleed. Die ontleding het getoon dat studente 'n absolute rekenstrategie gebruik het om oplossings vir die probleme te soek.

Die studie het die volgende beperkings:

1. Die studente was nie verteenwoordigend van sekondêre onderwysstudente in Suid Afrika nie. Slegs studente uit die ontwikkelinggroep was betrokke.
2. Die studente is nie willekeurig aan die eksperimentele en kontrole groepe toegewys nie. Hulle was in hul gewone kollege klasse.

Ondanks die bogenoemde beperkings, word daar aanbeveel dat:

1. Die probleem-gesentreerde benadering, wat die beginsels van uitkoms-gebaseerde onderwys ondersteun, behoort in die wiskunde kurrikulum vir skole en onderwyserskolleges gebruik te word.
2. Gepaste assesseringsmetodes, soos in die probleem-gesentreerde benadering gebruik, moet toegepas word.
3. Doeltreffende ondersteuningstelsels moet geïmplementeer word om onderwysers te ondersteun in hul oorgang na die probleem-gesentreerde benadering.

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CONTENTS

	PAGE
ACKNOWLEDGEMENTS	i
DECLARATION	ii
OPSOMMING	iii
SUMMARY	v
CHAPTER 1 INTRODUCTION	1
1.1 What is being dealt with?	1
1.2 Why is it a problem?	1
1.3 Overview	3
CHAPTER 2 LINEAR ALGEBRA IN SCHOOLS AND COLLEGES OF EDUCATIONS	4
2.1 Some comments on school mathematics in South Africa	4
2.2 Linear algebra in South African schools and colleges of Education	7
2.3 Historical ideas in teaching linear algebra	20
CHAPTER 3 SOME PROBLEMS IN MATHEMATICS EDUCATION	28
3.1 Some problems in mathematics learning	28
3.2 Some problems in mathematics education	31
3.3 Sense-making	33
3.4 Problem-centred approach	38

CHAPTER 4	METHODOLOGY AND ANALYSIS OF RESULTS	43
4.1	Motivation	43
4.2	Analysis	45
4.3	Discussion of results	48
4.4	Qualitative data	50
CHAPTER 5	DISCUSSION AND CONCLUSION	53
5.1	Discussion	53
5.1.1	Relatedness to Curriculum 2005	53
5.1.2	Assessment	57
5.1.3	Support system	59
5.2	Conclusion	59
APPENDIX	60
REFERENCES	64

Chapter 1

Introduction

1.1 What is being dealt with ?

Linear algebra has in one way or another, been a part of the curriculum in post-school education. Elements of linear algebra such as the solution of simultaneous linear equations and linear programming are also found in the school mathematics curriculum. Matrices, determinants and vectors are, however, being dealt with only in the post-school (Colleges, Technikons and Universities) phase. In colleges of education linear algebra is basically fashioned as an approach and sequencing as evident in textbooks from the United States of America. From experience these approaches and sequencing contribute towards students adopting a rote learning procedure, thus leading to the inability of students to deal with linear algebra in contextual situations. In order to address this problem, this study deals with ways in which mathematics students in a college of education handle contextually-driven, introductory linear algebra topics.

The problem-centred approach to mathematics has proven itself in the primary school situation to address the development of both conceptual and manipulative skills. This study is thus embedded within the problem-centred approach.

1.2 Why is it a problem ?

Dienes (1960,1964) stated the principle of multiple embodiments as an instrumental tool for enhancing the understanding of concepts and for retaining mathematical

structures. Several studies (e.g. Beardslee 1973; Gau 1973 cited in Katz 1995) investigated this principle as applied to concrete representation of concepts in elementary mathematics levels.

Algebraic system does not have easily visualised representations that describe their operations and relations. For example the students often make mistakes of the following nature:

$$2(2x+3)=4x+3.$$

It is reasonable to assume that similar misconceptions can result because of a manipulative drive in linear algebra. Such misconceptions are shown in the following example.

$$\begin{aligned} 2x+3y &= 6 \\ 3x+5y &= 14 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ 3 & 5 & 14 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$$

$$R_1 \times 3 \begin{bmatrix} 6 & 9 & 6 \\ 3 & 5 & 14 \end{bmatrix} \begin{matrix} R_3 \\ R_2 \end{matrix}$$

The misconception here is, instead of multiplying the left hand side and the right hand side, only the left hand side is multiplied by 3. When beginning students are presented with these systems, they encounter, probably for the first time, mathematical systems different in nature from the number system. They have difficulty in accepting the idea that a collection of numbers, such as a matrix, or a function, is a mathematical entity; namely, a mathematical object within a system that has its own structure and its own operations such as addition and scalar

multiplication of vectors. But still, any system used in the embodiment process would have no constructive cognitive effect if the situations being embodied were not familiar and understood by the student. For example, in solving problems in linear algebra many students have difficulty in moving from arithmetic to algebra. The arithmetic solution involves subtracting 2 and dividing by 5, the algebraic form $5x+2$ involves multiplication by 5 and addition of 2, which are really the inverse operations of division and subtraction.

1.3 Overview

The second chapter deals with the domain of linear algebra in schools and colleges of education as well as a historical preview on teaching linear algebra to indicate that there was some sort of connection between the problem-driven approach in dealing with linear algebra, particularly the solution of systems of equations and the way in which linear algebra was handled in the past. Chapter 3 deals mainly with some of the problems encountered in mathematics teaching and learning. Constructivism, different styles of sense-making and the problem-centred approach are also reviewed in this chapter. Chapter 4 elaborates on the data collection methods and the analyses of results. In this chapter I shall explain the methodology of sample and sampling, the design, i.e. the instrument (test) and the final results of the test. Thereafter the results and the conclusions drawn from the results will be discussed. The last part provides a qualitative analysis of students' work. The last chapter concludes with suggestions and issues for further research. Activities based on the problem-centred approach are included in the form of an appendix.

Chapter 2

Linear algebra in schools and colleges of education

This chapter reviews linear algebra as content within South African schools and colleges of education. The chapter starts with a reflection on school mathematics and some ideas related to it. It also deals with the occurrence of linear algebra in the school and college of education curricula. The next section deals with historical ideas related to linear algebra and its teaching.

2.1 Some comments on school mathematics In South Africa

The importance of science and mathematics education is profound. It is the foundation on which all science and technology development rests. These are key areas for the future of this country. "Mathematics is the gateway topic that opens up the rest" (National Research Council, 1989; cited in Ellis 1995; p.1). Currently in South Africa learners' progress in mathematics leaves much to be desired. The non-obtaining of matriculation exemption is largely due to learners not doing well in mathematics because a pass in mathematics is an important criterion for gaining a matriculation exemption. Furthermore, it came to the fore through the Third International Mathematics and Science Study (TIMSS) that South African grade 12 learners have tremendous problems with problem-solving activities in mathematics. Out of the twenty-one countries that participated in the TIMSS-study, South Africa was placed last (TIMSS, 1996). "Children are not interested in mathematics for its

own sake any more than they are interested in taxidermy for its own sake. They are interested in what relates to human purpose and principally in what relates to their own concern" (Fisher 1990; p.214 cited in Ellis 1995; p.10). The first point that needs to be made is that our present school mathematics syllabus is not constructed sufficiently of the problem solving. Some excellent teachers are introducing methods of teaching and assessment that aim to work on principles of the problem-centred approach, but time and again are prevented from doing so to any great extent by the syllabus, and particularly by the nature of the examination the students have to pass. The teachers frequently refer to the excitement pupils' experience, when they are engaged in problem solving activities. But they also express their frustrations in having to cut such activity short because of time constraints. The nature of the current examination hinders the development of a creative, exploratory attitude, which supposedly leads to deeper understanding.

Various educationalists in South Africa ascribe the problem in school mathematics performance to teachers' perception that learners should not perform well. A well-known South African scientist (Ellis 1995; p.4) describes the problem as follows. He happened to attend a standard 7 examiners' meeting, and found they were very worried by a particular question, which they felt, had gone wrong. He asked them what the problem was and was told:

Most of the pupils did very well on this question - they almost all got 80%!" So what should have been a cause for the rejoicing, the fruit of a highly successful teaching programme, was rather seen as a problem and clearly next year a more difficult question would

be set in order to reduce the success rate. Now they were in fact very nice and concerned people, but they were operating under the expectation that the pupils are not going to do well, and if they surprise you by in fact succeeding, then you adjust the goal posts to make sure that this does not happen again. By contrast, if one tries to introduce a mastery learning approach, you expect the pupils to do well; and then - without abandoning academic standards - you use methods that are based on the expectation that this can and will happen.

These kinds of issues occur even at the tertiary level. In the social science department the highest mark a student could score is just 75% to 80% even if the student is so good with his subject. Justification for that is "well, they didn't know everything there is to know about the subject. Yes, indeed - but they did everything a student at that level could have been expected to do! These top students should have been given between 95% and 100% - but it is customary to mark them down, the purpose of that missing 20% not being to say they could have done better, but rather to put across the message that their teacher knows more than they do.

It is within the context similar to the above whereby the teacher feels that he is the master of knowledge, which the study of pre-service teachers' handling of linear algebra activities was done.

Linear algebra is the topic of discussion in the next section.

2.2 Linear algebra in South African schools and colleges of education

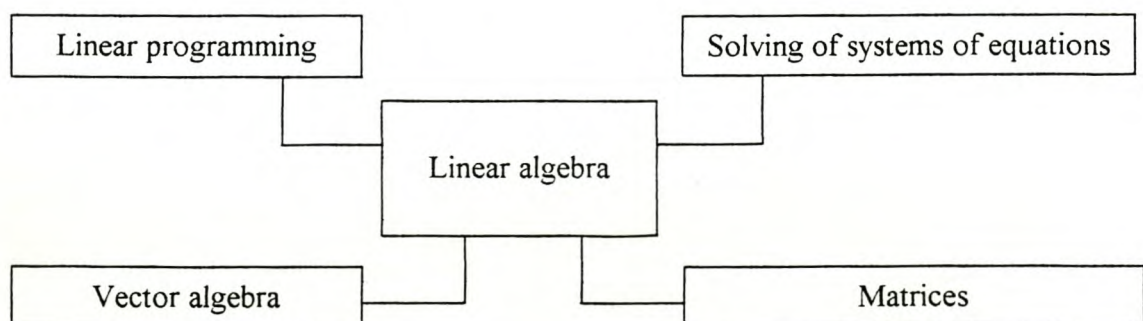
Although linear algebra can be linked to higher mathematical topics such as spectral theory, the approach adopted in this thesis is that followed by most proponents of elementary linear algebra. Halmos (1995:1), for example, asserts that

Linear algebra is concerned with several different kinds of operations (such as addition) on several different kinds of objects (not necessarily real numbers).

It is evident that the operational aspect of linear algebra is highlighted. A somewhat similar view is propagated by Strang (1980:1) in stating that

The central problem of linear algebra is the solution of simultaneous linear equations.

In other words linear algebra is the study of the system of linear equations and inequalities. This concentration on systems of linear equations has expanded as follows:



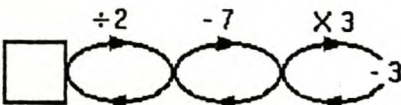
In South African schools the solution of systems of equations is restricted to two equations in two unknowns. It is interesting to note that the solution of systems of linear equations is not clearly indicated in the syllabus of college of education as linear algebra. Most teachers probably do not perceive their work with systems of linear equations as falling within the domain of linear algebra. The solution of systems of linear equations is essentially done in algebraic ways, similar to solving single linear equation. Some of the methods for solving single algebraic equations are:

- Arrow diagram
- Balancing
- Formal manipulations

The following are examples of the three methods

- (a) An example of an arrow diagram (Laridon, Brink, Burgess, Jawurek, Kitto, Myburgh, Pike, Rhodes-Houghton, Van Rooyen 1985). Use an arrow diagram to solve this number problem. Also write down the equation.

Sandy says that the number she is thinking of when divided by 2, decreased by 7 and then multiplied by 3, given a result of -3 . What number is she thinking of?



Her number is $[-3 \div 3] + 7] \times 2$

$$= [-1 + 7] \times 2 = 6 \times 2 = 12$$

The equation is $3\left(\frac{x}{2} - 7\right) = -3$

(b) Example of balancing

$$(2x + 3) = 5 \Rightarrow 2x + 3 - 3 = 5 - 3$$

$$2x = 2$$

$$x = 1$$

(c) Example of formal manipulation

$$2x + 3 = 5$$

$$2x = 5 - 3$$

$$x = \frac{2}{2}$$

$$x = 1$$

Similar methods are used in solving simultaneous equations. For example systems of linear equations can be dealt with numerically, graphically or algebraically.

(a) Example of numerical solution

$$2x + 3y = 3$$

$$x - y = 4$$

x	Y	L.H.S 1	L.H.S 2	R.H S 1	R.H.S 2
1	0	2	1	3	4
0	1	3	-1	3	4
3	-1	3	4	3	4

From the table it is clear that when $x = 3$ and $y = -1$, the left-hand sides of the equations are equal to the right hand sides. With this method one can see it is tedious, but it highlights what is meant by a solution of simultaneous equations.

(b) Example of graphical solution

$$2x + 3y = 3$$

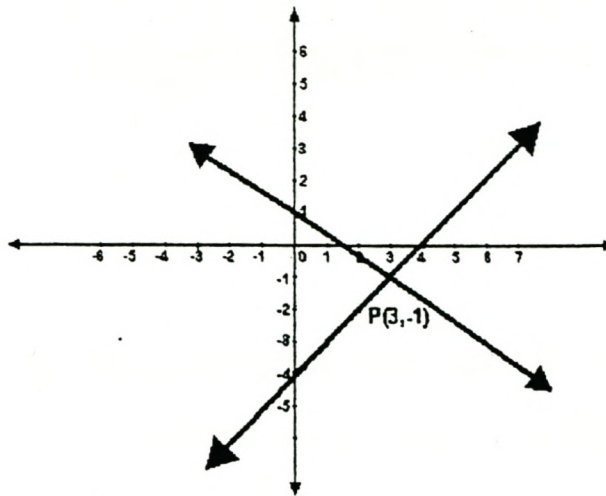
$$x - y = 4$$

Tabulate: $2x + 3y = 3$

x	0	$1\frac{1}{2}$	3
y	1	0	-1

$$x - y = 4$$

x	0	4	2
y	-4	0	-2



The solution is at $P(3, -1)$

The graphic calculator as a new technology fits well with the graphical solution approach. Also if two equations such as $y - 2x = 4$ and $3y + 2 = 6x$ are graphed, it gives a sense of the no solution situation, which the algebraic method seems not to concentrate on and can be tedious using the numerical approach.

(c) Example

$$2x + 3y = (1)$$

$$x - y = 4 \quad (2) \times 3$$

$$3x - 3y = 12$$

$$2x + 3y = 3$$

$$5x = 15$$

$$x = \frac{15}{5} = 3$$

Substitute in equation 2

$$3 - y = 4$$

$$-y = 4 - 3 = 1$$

$$y = -1$$

Another method is equalisation of coefficients (elimination).

This method is equivalent to manipulation with substitution and in a sense a true elimination (Gaussian) method. At matric level the topic linear programming is meant for Higher Grade only. Furthermore, it is entirely new to the 1983 syllabus. Since linear programming is based on the idea that a decision maker wants to maximise an objective or to minimise an objective, it has many real-life applications, but sadly most of the time, the question asked in an examination paper will end up being a mental sum. For example:

An appliance manufacturer makes two kinds of ovens: model A, which earns R10 profit, and model B which earns R12 profit. Each month the manufacturer can produce up to 600 units of model A and up to 500 units of model B. If there are only enough man-hours available to produce no more than a total of 900 ovens a month, how many of each kind should be produced to obtain the maximum profit? (UNISA. FDE – JANUARY 1998).

It is obvious that the answer is 400 and 500. The student thus can solve the problem without paying attention to the salient ideas of linear programming. The problem-centred approach attempts to engage students in the salient aspects of a mathematical topic.

Linear algebra in colleges of education

At colleges of education the following aspects of linear algebra are dealt with in first year level.

1. Systems of linear equations in two unknowns.
 - 1.1 Graphical and algebraic solution of systems of linear equations.
 - 1.2 Application of systems of linear equations in the solution of problems.
2. Linear Programming.
 - 2.1 Graphical representation of $ax+by+c=0$.
 - 2.2 Application in linear programming (only graphically).

To the greatest surprise we hardly ever see any linear algebra at all in the second year level. In course three:

1. Systems of linear equations.
 - 1.1 Solution of systems of linear equations using the Gaussian elimination method with a maximum of three equations in three unknowns.

For example:

Solve the following system of equations.

$$x + y - z = 9 \quad (1)$$

$$x + y - z = 1 \quad (2)$$

$$x - y + z = 3 \quad (3)$$

Row operation

$$\begin{pmatrix} 1 & 1 & 1 & 9 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -2 & 0 & -6 \\ 0 & 0 & -2 & -8 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \quad \begin{matrix} R_4 : R_2 - R_1. \text{ Delete } R_2 \\ R_5 : R_3 - R_1. \text{ Delete } R_3 \end{matrix}$$

From row-echelon form go back to the equivalent system of equation.

$$x + y + z = 9 \quad (1)$$

$$-2y + 0 = -6 \quad (2)$$

$$-2z = -8 \quad (3)$$

$$z = \frac{-8}{-2} = 4$$

$$2y = -6$$

$$y = \frac{-6}{-2} = 3$$

Substitute into (1) : $x = 2$

$$\therefore (x, y, z) = (2, 3, 4)$$

Immediately after that, vector algebra has been introduced, without making any connections or even mentioning that it is a part of linear algebra. In vector algebra the following sub-topics have been covered.

2. Vector algebra.

2.1 Definition of a vector.

2.2 Geometrical representation of a vector.

2.3 Vector addition by the parallelogram method using trigonometry and by composition of components.

2.4 Operations – sum and difference; multiplication by a scalar and inner product.

2.5 The vector equation Cartesian equation of a line.

Vector is defined as a physical quantity with both size (magnitude) and direction.

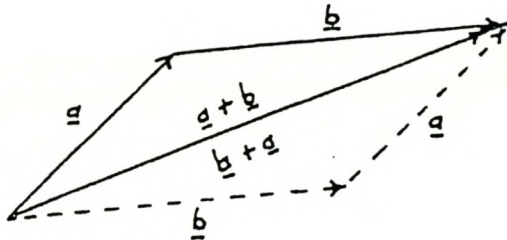
Geometrical representation of a vector

An arrow represents a vector. The length of the arrow is its magnitude and the direction of the arrow is the direction of the vector.

Additions of vectors are handled algebraically and geometrically in the following way.

Addition of vectors – Geometrically

If \vec{a} and \vec{b} are vectors, $\vec{a} + \vec{b}$ is defined to be the diagonal of the parallelogram spanned by \vec{a} and \vec{b} . Position \vec{b} so that its initial point coincides with the terminal point of \vec{a} .



Algebraically

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ (1, 2) + (2, 4) &= (3, 6)\end{aligned}$$

In Matrices, the following topics have been covered.

- 3. Matrices.
 - 3.1 Definition.
 - 3.2 Equality of matrices.
 - 3.3 Sum of matrices.
 - 3.4 Product of matrices.
 - 3.5 Product of matrices.
 - 3.6 Unit matrix.
 - 3.7 Inverse of a matrix.
 - 3.8 Practical application.

From our experience when dealing with practical application, our students at colleges of education are really finding it difficult to formulate the respective equations.

Example:

A veterinarian wants to control the diet of an animal so that on a monthly basis the animal consumes (besides, hay, grass and water) 60 pounds of oats, 75 pounds of corn, and 55 pounds of soybeans. The veterinarian has three feeds available, each consisting of oats, corns and soybeans as shown in the table. How many pounds of each feed should be used to obtain the desired mix? (Sobel & Lerner 1995; p.554).

	Oats	Corn	Soybeans
1lb of feed A	6 OZ	5 OZ	5 OZ
1lb of feed B	6 OZ	6 OZ	4 OZ
1lb of feed C	4 OZ	7 OZ	5 OZ

Solution: Let x = pounds of feed A

y = pounds of feed B

z = pounds of feed C

Then $6x$ = ounces of oats in x pounds of feed A

$6y$ = ounces of oats in y pounds of feed B

$4z$ = ounces of oats in z pounds of feed C.

The total number of pounds of oats required is 60.

In ounces this is $60 \times 16 = 960$.

Thus $6x + 6y + 4z = 960$

Analysis for the total ounces of corn and soybeans leads to this linear system.

$$6x + 6y + 4z = 960 \quad (\text{total ounces of oats})$$

$$5x + 6y + 7z = 1200 \quad (\text{total ounces of corn})$$

$$5x + 4y + 5z = 880 \quad (\text{total ounces of soybeans})$$

Now proceed with the matrix method to solve the system.

$$\begin{bmatrix} 6 & 6 & 4 & 960 \\ 5 & 6 & 7 & 1200 \\ 5 & 4 & 5 & 880 \end{bmatrix} \leftarrow \text{argument matrix of the system.}$$

$$\begin{bmatrix} 1 & 1 & \frac{2}{3} & 160 \\ 5 & 6 & 7 & 1200 \\ 5 & 4 & 5 & 880 \end{bmatrix} \leftarrow \frac{1}{6} \times (\text{row1})$$

$$\begin{bmatrix} 1 & 1 & \frac{2}{3} & 160 \\ 0 & 1 & \frac{11}{3} & 400 \\ 0 & -1 & \frac{3}{5} & 80 \end{bmatrix} \leftarrow -5 \times (\text{row 1}) + \text{row 2}$$

$$\begin{bmatrix} 1 & 1 & \frac{2}{3} & 160 \\ 0 & 1 & \frac{11}{3} & 400 \\ 0 & 0 & \frac{16}{3} & 80 \end{bmatrix} \leftarrow 1 \times (\text{row2}) + \text{row3}$$

Converting into the equivalent triangular system and using back-substitution gives $z=90$, $y=70$, and $x=30$. Therefore, 30 pounds of feed A, 70 pounds of feed B, and 90 pounds of feed c should be combined to obtain the desired mix.

This matrix procedure also reveals when a linear system has no solutions, that is, when it is an inconsistent system. In such a case we will obtain a row in a matrix of the form. $000 \dots 0 | P$ Where $p \neq 0$, but when this row is converted to an equation, we get the false statement $0=p$.

Over and above all those topics like matrices, vector algebra and linear programming have been in the third year level, no connections between these topics have been mentioned. Furthermore, it is not even indicated as the part of linear algebra in the syllabus.

It appears that there was some connection and a form of problem-driven approach with the dealing of linear algebra, particularly the solution of system of equations during the historical development of linear algebra. This is the emphasis of the next section.

2.3 Historical ideas in teaching linear algebra

As a subject in the undergraduate curriculum at American universities, linear algebra is relatively new (Katz 1995; p.189). Though many of the individual topics now included under the name linear algebra have roots stretching back many centuries, their organization into a program of study is only a few decades old. Perhaps because of this, the typical structuring of a linear algebra course follows the historical order of development more closely than that of many other courses studied by undergraduates. For example, the first linear algebra topic usually encountered is that of the solution of system of linear equations, a subject that to some extent was studied by the Babylonians nearly 2000 years ago. The next topic may well be determinants, which date from roughly 300 years ago, and the elements of vector geometry in 2-space and 3-space, a concern of the early nineteenth century. The mere abstract notions of vector space and linear transformations, built upon concrete foundations, were not fully developed in the mathematical literature until the late nineteenth and early twentieth centuries and are typically studied toward the end of a linear algebra program.

Systems of linear equations

We begin by considering the ancient methods of solving systems of linear equations. Chinese scholars over two millennia ago developed a method for solving systems of two equations in two unknowns, which start with the “guessing” of possible solutions and then concludes by adjusting the guess to get the correct solution. This shows clearly that the Chinese understood the basic idea of linearity (Katz 1995; p.201).

The Chinese method occurs in chapter 7 of the Jinzhang Shanshu. The basic idea is as follows: suppose one is given as a function of one variable, say $f(x) = rx = s$, and wants to solve the equations $f(x) = b$. It may be that b is not known. The Chinese tried two values for the unknown, say x_1 and x_2 and determine what they called surplus b_1 and the values of the unknown, say x_1 and x_2 , and determined by what they called the surplus b_1 and the deficiency b_2 .

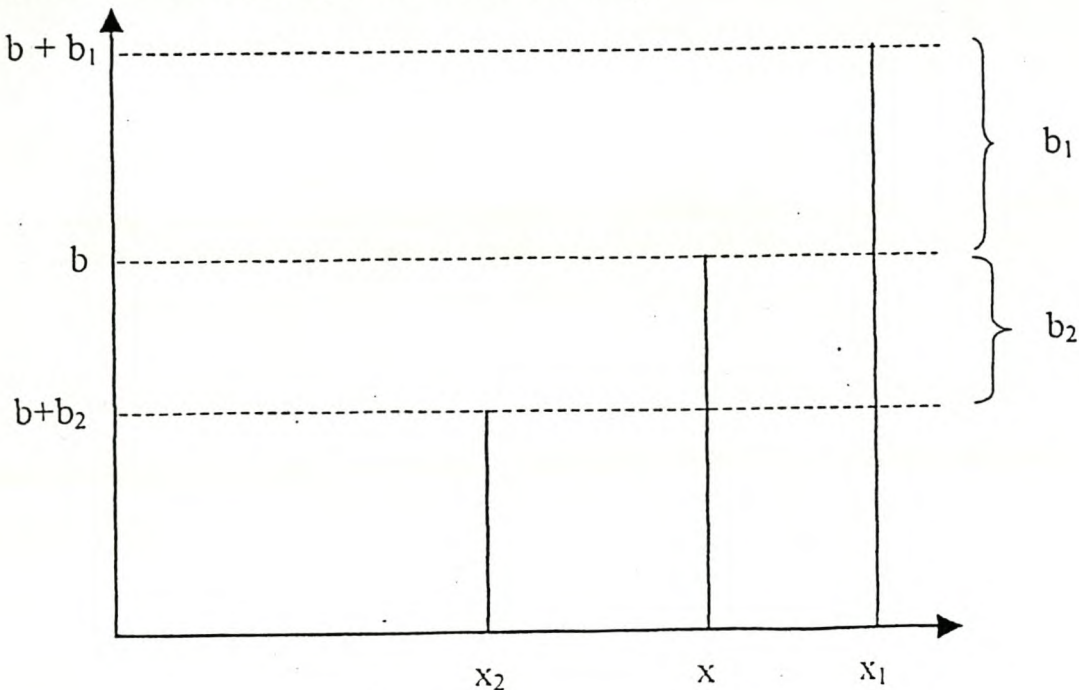


Fig 1

That is, b_1 and b_2 were determined so that $f(x_1) = b + b_1$ and $f(x_2) = b$.

To find the desired value of x , one uses linearity.

$$\frac{b_1}{x_1 - x} = \frac{b_2}{x - x_2}$$

A teacher can easily explain this proportion to a class by the use of a straight line

(Figure 1). It then follows that $b_1 x - b_1 x_2 = b_2 x_1 - b_2 x$ or that

$$x = \frac{b_1 x_2 + b_2 x_1}{b_1 + b_2}$$

It is doubtful that the Chinese discovered the rule by using the diagram (Figure 1), but they obviously understood the consequence of linearity.

Here are two examples of the rule taken from the Jinzhang.

one pint of good wine costs 50 gold pieces, while one pint of poor wine costs 10. Two pints of wine are bought for 30 gold pieces. How much of each kind of wine was bought? (Katz 1995; p.189).

This problem can be set up as a system of two equations in two unknowns:

$$x + y = 2$$

$$50x + 10y = 30$$

It is the left side of the second equation, with $2 - x$ substituted for y , which is $f(x)$.

That $f(x) = 40x + 20$. The two trial values are $x = \frac{1}{2}$ for which $f(\frac{1}{2}) = 30 + 10$, and

$x = \frac{1}{5}$ which $f(\frac{1}{5}) = 30 - 2$. Hence $b_1 = 10$, $b_2 = 2$, and the formula above, $x = \frac{1}{4}$ is the number of pints of good wine purchased. The value for the number of pints of poor wine, $y = 1\frac{3}{4}$ is then easily calculated. A second example shows that it is not necessary to know b :

Several people buy some chickens in common. If each paid 9 coins, the surplus is 11; if each paid 6, the deficiency is 16. How many people were there and what is the price of the chickens? (Katz 1995; p.190).

In this case, let x represent the number of coins paid by each person. Then $f(x) = rx + s$ represent the total price paid, where r represents the number of people. The two trial values are $x = 9$, for which $b_1 = 11$, and $x = 6$, for which $b_2 = 16$.

Again, the formula above gives $x = \frac{210}{27} = \frac{70}{9}$ as the price paid by each. The smallest integral solution for the question that there were 9 people and the price was 70 coins, is easily checked to be correct.

The Chinese also developed the essence of what is today known as the method of Gaussian elimination to solve larger systems. Though as usual they did not provide any reason for their method, the important idea to convey from this method is that, we have a system in which one of the equations only contains one unknown, a second only two, etc. Though it is extremely unlikely that Gauss was influenced by the Chinese in his development of Gaussian elimination, it is important to note that, in

fact, the Chinese had discovered the method close to two millennia before the “Prince of Mathematics”.

Another Chinese example from chapter 8 of the nine chapters:

There are three classes of corn, of which three bundles of the first class, two of the second and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the two of the second and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class? (Katz 1995; p.191).

This problem can be translated into a system of 3 equations in 3 unknowns:

$$3x + 2y + z = 39$$

$$2x + 3y + z = 34$$

$$x + 2y + 3z = 26$$

Using counting rods, the Chinese physically set up this problem on their counting board in the configuration of a matrix according to the direction.

Arrange the 3, 2 and 1 bundles of the three classes and the 39 measures of their grains at the right. Arrange the other conditions at the middle and at the left.

1 2 3

1 2 2

2 1 1

26 34 39

The author then instructs the reader to multiply the middle column by 3 and subtract off twice the right column, then to multiply the left column by 3 and subtract off the right column. This reduces the matrix to

$$\begin{array}{ccc} 0 & 0 & 0 \\ 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{array}$$

$$$$

$$$$

$$$$

The next step is to multiply the left column by 5 and then subtract 4 times the middle column, leaving a “column-reduced” matrix.

$$\begin{array}{ccc} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{array}$$

$$$$

$$$$

$$$$

This is the equivalent of the triangular system:

$$3x + 2y + z = 39$$

$$5y + 2z = 24$$

$$36z = 99$$

from which one easily finds $z = \frac{99}{36} = 2\frac{3}{4}$ and then y and x by back-substitution. In

South Africa we adopt the horizontal approach.

For example:

$$3x + 2y + 2z = 39$$

$$2x + 3y + z = 34$$

$$x + 2y + 3z = 26$$

$$\begin{bmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 26 \\ 2 & 3 & 1 & 34 \\ 3 & 2 & 1 & 39 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 26 \\ 0 & -1 & -5 & -18 \\ 0 & -4 & -8 & -39 \end{bmatrix} \begin{array}{l} -2r_1 + r_2 \\ -3r_1 + r_3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 26 \\ 0 & 1 & 5 & 18 \\ 0 & 0 & 12 & 33 \end{bmatrix} -r_2$$

$$x + 2y + 3z = 26$$

$$y + 5z = 18$$

$$12z = 33$$

$$z = \frac{33}{12} = 2\frac{3}{4}$$

$$y + 5(2\frac{3}{4}) = 18$$

$$y = 18 - 13\frac{3}{4} = 4\frac{1}{4}$$

$$x + 2(4\frac{1}{4}) + 3(2\frac{3}{4}) = 26$$

$$x = 9\frac{1}{4}$$

In this Chapter, I have discussed linear algebra in South African schools and colleges of education. I have also discussed some historical ideas related to linear algebra and its teaching. What emerges from the historical development is that problems played a central role. The problem-centred approach is based on learning by solving problems. This approach is included in this study and is discussed in the next chapter.

Chapter 3

Some problems in mathematics education

This chapter elaborates on some problems associated with mathematics education. Constructivism as a theory of learning is reviewed together with sense-making and the different types of sense-making. The chapter concludes with a discussion of the problem-centred approach.

3.1 Some problems in mathematics learning

This study deals with elementary linear algebra. As alluded to in the previous chapter the linear algebra ideas with which students enter colleges of education are mainly those encountered in secondary school algebra. One of the problems students experience in secondary school algebra is that of symbolization. A definition of a mathematical object consists of variables. Some of these variables are encoded in the symbol of the object; others are not (Katz 1995). Those, which are not encoded in the symbols, are, usually fixed throughout the discussion of the object. Textbooks in linear algebra lack uniformity in this aspect of symbolization. In some text books the distributive law is represented as $a(x+y)=ax+ay$, while in some other texts, it is represented as $a(b+c)=ab+ac$. It is likely that a symbol is better understood and remembered if it expresses the main and salient variables in the concept it represents. On the other hand, a symbol that excludes non-fixed variables or includes superfluous variables would not correspond adequately to the meaning of

the concept it represents, and consequently it becomes difficult to encode and to recognise (Katz 1995). Algebraic symbols do not speak for themselves. What one actually sees in them depends on the requirements of the problem to which they are applied. When one looks at an algebraic expression such as $3(x+5)+1$ what does one see? It depends. In certain situations one will probably say that this is a concise description of a computational process. $3(x+5)+1$ will be seen as a sequence of instructions. In another setting one may feel differently. $3(x+5)+1$ represents a certain number. If the context changes, $3(x+5)+1$ may become yet another thing: a function – a mapping that translates every number x into another (Sfard & Linchevski 1994; p.191).

The distinction between operational and structural conception, which we made in the opening example, shows what is going on in people's heads rather than to what they actually communicate to the world by means of written records. In other words there is a deep ontological, i.e. the nature of existence, gap between operational and structural conceptions. Certain kinds of algebra are operational in their character while others are structural. Words are not manipulable in the way symbols are. It is this manipulability, which makes it possible for algebraic concepts to have the object-like quality.

Example:

In a certain college there are six times as many students as there are professors. Use S for the students and P for the number of professors to write an equation for the situation (Olivier 1992).

The research shows (Olivier 1984) that 58% of standard 8 pupils responded with $P=6S$ which means they are interpreting S as an abbreviation for students and P for professors. In essence, they are using letter symbols as labels or as abbreviations for units as in 6 gram = 6g. Pupils often carry this prior meaning into algebra, with disastrous results, as the example shows. This misconception of the meaning of letter symbols in algebra is reinforced when teachers treat $2x+3x$ as mere abstract “letters” and $a+b$ as apples and bananas. Pupils need to construct meaning for letters as numerical variables in order to cope with algebra.

Davis (1984) offers an alternative explanation for the students-professor error. He suggests that pupils may indeed have a numerical variable schema available. When children give wrong numbers it is not so often that they are wrong, as that they are answering a different question. If we want to account for pupils’ misconceptions, we must look at pupils’ current schemas and how they interact with previous schemas, with instruction and experience (Olivier 1992; p.197).

Let us consider the following problem:

When 3 is added to 5 times a certain number, the sum is 43. Find the number.

In algebraic language it is written as:

$$5x+3=43$$

Then we simplify as:

$$5x+3-3=43-3$$

When we simplify, $5x=40$. Thus $x=8$. The “certain number” in the problem is 8. In solving problems of this kind, many students have difficulty in moving from arithmetic to algebra. On the other hand the arithmetic solution involves subtracting 3 and dividing by 5, the algebraic form $5x+3$ involves multiplication by 5 and addition of 3, which are the inverse operations. To set up the equation, you must think exactly the opposite of the way you would solve it using arithmetic (Usiskin 1996; p.12). So the teacher’s job is to find out what questions they are in fact answering. The teacher must help pupils to differentiate between such cases and stress the conditions under which such cases are applicable. If we understand the general principles of cognitive functioning from a constructivist perspective, which I shall discuss in the next section, we will realise that, for the most part, children do not make mistakes because they are stupid but their mistakes are rational and meaningful efforts to cope with mathematics. Of course, these derivations like $P = 6S$ is objectively illogical and wrong, but psychologically, from the child’s perspective they make a lot of sense. (Ginsburg, 1977 cited in Olivier 1992).

3.2 The problems in mathematics education

Confrey (1996; p.477) stated the following reasons for some of the problems that we are experiencing in mathematics education:

1. Student's mathematical knowledge is limited and rigid - they focus on answer, they expect whole number solutions.
2. Mathematics is classified as formal knowledge and hence is isolated from common experience and common sense-making.
3. Students rely on external sources of authority to evaluate their competence in mathematics.

4. Students are alienated from mathematics and experience fear, avoidance, anger and apathy towards it.

Difficulties in mathematics learning can be classified even further into two categories:

1. Students' conceptions of mathematics.
2. Students' beliefs about how mathematics is to be learned.

A theory about knowledge acquisition offered to guide this classroom reconstruction is constructivism. The theory of constructivism is a theory about the limits of knowledge. Knowledge consists of that which we can know. What is knowable is necessarily a product of our own mental acts or constructions (Confrey 1996; p.477).

A person's knowledge is necessarily the product of her/his own construction. If a person chooses to believe in an objective reality, that belief is an act of faith. Knowledge consists of a mental action on the world; it is a way of seeing, of organising experience.

The implication for mental action of such a theory may be made clearer by the following example:

Introducing the idea of "symmetry" - show the pupils a fern and folds it in half to show how the two sides "landed on top of each other". Doing so entailed an action; in this case a physical action. Then we can speak about the axis of symmetry, point it out as though it were in the plant. The axis of symmetry is a label for a mental construct

involved in the action of folding the plant physically or mentally (Sfard & Linchvski 1994).

The constructivist view seems to imply that in teaching emphasis has to be shifted from telling to facilitating learning; teaching by imposition has to be replaced by teaching by negotiation. Learning is considered as a two-directional flow of information between students and teachers.

Constructivism is neither an approach to teaching nor does it favour any particular way of learning, it merely claims that all knowledge is personally constructed in social contexts (Human 1996; p.15). Constructivism encourages dialogue between the students and teachers with a view to providing an opportunity for the child to construct his/her knowledge. Learning activities, based on the above principles, should thus allow for:

- a) discussion between pupils.
- b) discussion between pupils and teachers.
- c) learners to construct their own knowledge (Moodley et al- 1992; p.36).

3.3 Sense-making

Mathematics classrooms are dominated by instruction and performance of rote procedures " to get the right answer". A primary goal for the study of mathematics is to give children experiences that promote the ability to solve problems and to build mathematics from situations generated within the context of everyday experiences. Students are also expected to make conjectures and conclusions and to discuss their

reasoning in words, both written and spoken. Moreover, students learn to value mathematics when they make connections between concepts and skills and between mathematics and other areas in the curriculum.

Representation of knowledge

1. Syntactic categorisation: The nature of the difference between mathematical definitions, methods, propositions, concepts, procedures etc. can be regarded as syntactical in very much the same way as components of language (e.g. verbs, adverbs, adjectives, full sentences are different).

Miscategorisations are often observed in learners, e.g. not to perceive letters used in algebra as symbols for referents but as objects in their own right, referring to one triangle as being "a congruent triangle", failing to observe that an algebraic manipulation process (e.g. factorisation on "adding polynomials" leads to a proposition).

2. Syntactic analysis: One has comprehended some structure in a given representation, or one has had difficulty in comprehending the structure. The information that results from syntactic analysis is syntactic information, and more specifically procedural, propositional, conceptual information.
3. Assigning meanings: Assign meaning to some of the terms used in the above representation. On the other hand, one may not have done so, experiencing some sense of being lost as to the possible meaning of terms.

4. Rationalisation: One may have decided that some specific purpose may be accomplished by applying the rule in specific instances or one may not have sensed any purpose that can be served. Alternatively or in addition, one may sense that if the representation was part of some compulsory course of training, one may try to meet certain scholastic demands and/or satisfying the expectations of a teacher or a sponsor of one's studies. In other words rationalisation in the sense of assigning, seeking, experiencing a rationale or purpose to or experiencing that there is no rationale that makes sense to one, seems to be an appropriate label.
5. Epistemological categorisation: One may have several perceptions on certain rules in mathematics. For example, how it is formulated, one may have perceptions that it was what some mathematicians decided certain terms to mean, like some sort of agreement or convention. Alternatively, one may have the perception that somebody discovered or hypothesised certain terms. A third possibility is that one perceived it as a specific plan or method designed.
6. Justification or validation: One probably gave some thought to the issue of whether the values are true, whether it actually works in the sense that it achieves the purpose one has in one's mind, and one has either made some judgement about this or one has reserved judgement. One has reasons for judging that it actually works, or doubting that it works, or reserving judgement.

7. Relating mathematically: One may or may not relate certain rules to some mathematical ideas, one already had. One most probably, at least tried it.

The information produced by the last three actions may be labelled epistemological, justificational and relational information respectively. We sometimes find it convenient to refer to the process of acquisition of mathematical knowledge as imbedding syntactic information. All these actions are performed in some form or another whenever a person acquires mathematical knowledge. It is these actions collectively that we will refer to as constructing knowledge.

One person cannot assign meaning for another. One person can provide another with a representation of a meaning, but to comprehend this representation the "receiver" had to make the assignment in his/her own mind too (Human 1996;p.14).

Many variations of syntactic categorisation and analysis, assigned meaning, experienced purposes, perceived justification and relations and epistemological categorisation exist for the "same" mathematical idea. Let us consider the following example:

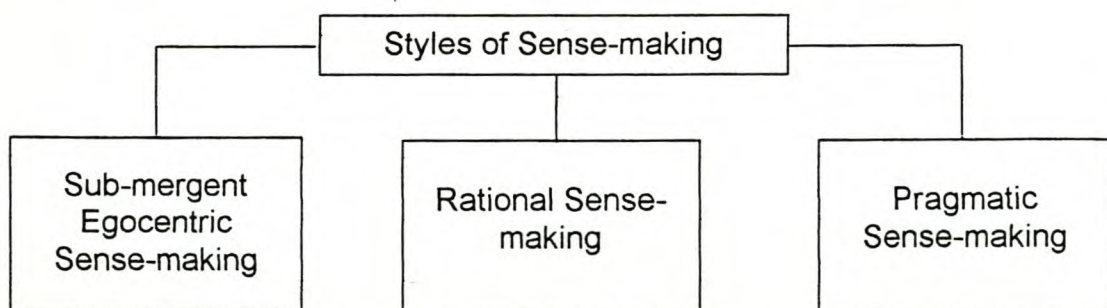
"To remove brackets in an algebraic expression, the term outside the bracket is multiplied with each of the terms inside the brackets".

A host of different meanings may be used for the notion of an algebraic expression, the notion "term" and the letter symbol used in an algebraic expression. One possibility is that the letter symbols are interpreted as indicating numerical variables.

Alternatively, letter symbols may be interpreted simply as "objects" and not as symbols at all.(e.g. apples and pears). The result of removing brackets (e.g. the expression $6x+10$ as the result of removing the brackets in $2[3x+5]$) may be interpreted as an expression equivalent to the original expression. In the later case it may be interpreted as the answer produced by executing a certain procedure. The information may or may not be related to previous numerical experience and the intuitive knowledge that multiplication distributes over addition.

The quality of knowledge

The question of the quality (worth, value, merits) of different embodiments of mathematical information arises. One possibility is to relate the quality of knowledge to the authenticity of embodiments in the degree of correspondence of personal embodiments of mathematical information with those of other people who make real sense of mathematics, including sensibly practising mathematicians (Human 1996; p.8). Different styles of sense-making is summarised in the following diagram.



3.4 The problem-centred approach

Problem-centred approaches as the end of learning path

The problem-centred approach is a particular strategy to empower learners to rationally make sense of mathematics, to prevent the learners from submergent meanings, from imbedding mathematical information in guiding the way in which learners process mathematical information. For this to happen, it is essential that the learners really understand the problem used as contexts. The problem-centred approach does not merely involve the contextualisation of mathematics to be learnt in problematic contexts, it also involves that learners engage in attacking problems prior to having constructed the target knowledge. In this sense the problem-centred approach differs radically from the well-known strategy of the teacher stating a problem and announcing that mathematics needs to solve the problem, and the students, proceed with the exhibition (Human 1993; p14).

Classroom culture in problem-centred learning

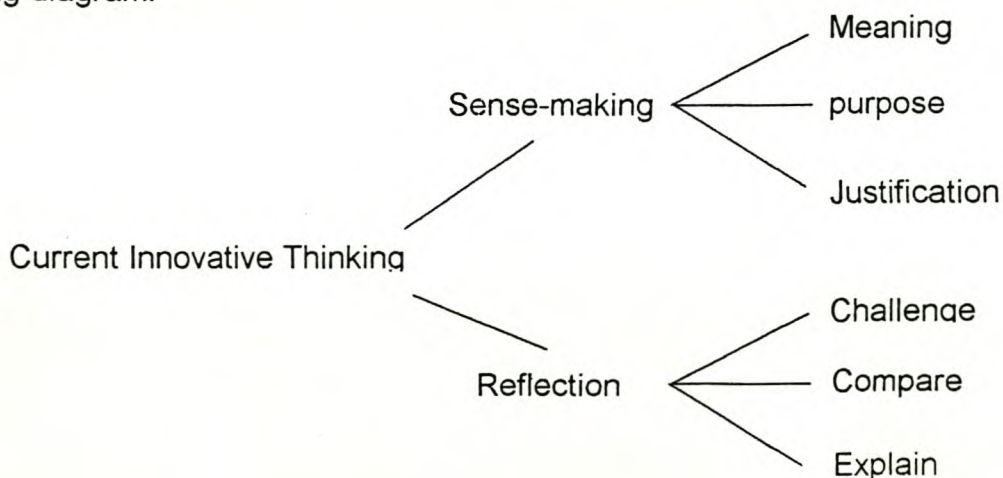
Some salient characteristics of a problem-centred learning classroom include (Murray, Olivier & Human 1993):

1. Students are presented with problems that are meaningful and interesting to them, but which they cannot solve with ease using routinized procedures or drilled responses.

2. The teacher does not demonstrate a solution method, nor does she steer any activity (e.g. questions or discussion) in a direction that she had previously conceived as desirable, yet she expects every student to become involved with the problem and to attempt to solve it. Students' own invented methods are expected and encouraged.
3. It is expected of students to discuss, critique, explain and when necessary, justify their interpretations and solutions.

However, circumscribe the teacher's role to that of facilitator or chairperson, or if necessary, devils-advocate and regard students' mathematical discourse as the main vehicle for learning. We regard this type of discourse as compatible with what Richards (1991; p.15 cited in Murray et al 1993) calls inquiry mathematics "asking mathematical questions, solving mathematical problems that are new to you, proposing conjectures, listening to mathematical arguments.

The main features of the problem-centred approach can be summarised in the following diagram.



Purpose of social-interaction in problem-centred classrooms

1. Social interaction creates opportunities for students to talk about their thinking and this talk encourages reflection.
2. Students learn, and learn from each other, by listening to and trying to make sense of other procedures and concepts being explained.
3. Through classroom social interaction the teacher and students construct a consensual domain of taken-to-be-shared mathematical knowledge that both makes possible communication about mathematics and serve to constrain individual student mathematical activity (Murray et al 1993).

According to the Cockroft Report (1982 cited in Moodley et al 1992; p. 68) "The ability to solve problems is at the heart of mathematics".

A class has just learnt to solve quadratic equations using the method of factorisation.

Exercise 1: Solve (Nicholson, cited in Moodley et al 1992; p.68)

1 $x^2 - 7x + 12 = 0$

2. $x^2 - 9x + 20 = 0$

3. $x^2 + x - 20 = 0$ etc

In the above questions, if the method of solution is known and if correctly applied the correct values of x will be found. The pupils do not have to discover a course of action, or think about how to get over an obstacle. Some pupils may find this difficult, but this would be because of a lack of understanding or the inability to factorise trinomials.

Another class has been taught how to solve by factorisation and is now ready to proceed to find out which equations do not factorise. The following is given to them as an introductory exercise:

Exercise: Solve

1. $x^2 = 4$
2. $(x^2 + 1)^2 = 4$
3. $(x^2 + 1)^2 - 3 = 13$
4. $x^2 + 2x - 7 = 0$

One of the strategies of a problem-centred approach is to look for a simpler problem of a similar nature that can be solved.

Other facets of a problem-centred approach are independent thought, an understanding of the problem and a degree of creativity in finding a solution.

The teacher must become a manager, a questioner, a provider of resources and a guide. The teacher must challenge at the right moment, get pupils to evaluate their

solution and to ask more questions of a particular situation. Pupils should be encouraged to be creative and to try various approaches to a problem. Perhaps pupils could work in pairs or in groups, there should be free discussion and the classroom organisation should be suitable for them. Solutions should be challenged by peers and alternative solutions sought and investigated.

In this chapter I have discussed some possible misconceptions that students can make in algebra. Then I have tried to identify some problems in mathematics education and solutions to these problems from the constructivist's point of view. In the last part I have included sense-making and the problem-centred approach.

Chapter 4

Methodology and analysis of results

This section of the study deals with the prospective teachers' development of solution procedures in linear algebra. The central questions guiding the study were: Would an emphasis on the problem-centred approach lead students to a better understanding of the solution of systems of linear equations? Are they able to cope with the current innovative thinking? The analysis of the results will give us more ideas about that. By understanding the solution of systems of linear equations we mean the ability to determine meaning, purpose and justification.

In this chapter I shall explain the methodology that I have used. The next section provides the analysis of the result of the experiment and the conclusions that I have drawn from the result. The last section incorporates an analysis of students' work.

4.1 Motivation

The primary concern of the study is how pre-service teachers perform after they have been exposed to a section of a linear algebra course based on the problem-centred approach. The performance or achievement throughout the year could not give us a true reflection of the problem-centred approach.

The best way to assess superior performance is a simple experimental design comparing the achievement of an experimental and a control group on regular tests in linear algebra.

Methodology

The students who participated in this study were 120 third year students enrolled in Secondary Teacher's Diploma course. All students had the same formal education in mathematics. We had two groups of 60 each. One group of 60 were exposed to a problem-centred approach in linear algebra. The students in the experimental group were presented with linear algebra problems which could be set to foster the ideals inherent in a problem-centred approach, but which they cannot solve with ease using routine procedures or drill responses. The activities attached as appendix were mainly used for this purpose. The teacher did not demonstrate a solution method, nor did she steer any activity. Questions and discussion in a certain direction were welcomed but still I expected every student to become involved with the problem and to attempt to solve it. Students' own invented methods were expected and encouraged. The teacher expected the students to discuss, critique, explain and when necessary, justify their interpretations and solutions.

Since some of the work that people do require some degree of co-operation and communication with others, the method used in an introductory problem-centred approach is co-operative learning. There are numerous variables, which interact to influence the processes of an outcome of a co-operative learning experiment. A number of these variables, such as academic achievement, time spent on group activities and skill or personal characteristics are directly observed in the experimental group. In order to gain an encompassing view and broad insight into the co-operative learning experience, a comprehensive evaluation strategy is needed. A case-study approach, which used oral class discussion, test and group interviews

were used. It should be noted that although this study used co-operative techniques co-operative learning was not the objective of investigation.

The control group never had any opportunities of experiencing the problem-centred approach. They were exposed to the traditional teaching style whereby the teacher states the problem and announces that mathematics needed to solve the problem, and proceeds with the exhibition.

Both the groups, the experimental group and the control group, were given the same test. The items in the test were chosen to be simple problems on linear algebra.

Problem 1: Danial has R575 in one-rand, five-rand, and ten-rand notes. Altogether he has 95 notes. The number of one-rand notes plus number of ten-rand notes is five more than twice the number of five-rand notes. How many of each type of bill does he have?

Problem 2: The sum of three numbers 33. The largest is number one less than twice the smallest number. Three times the smallest number is one less than the sum of the other two numbers. Find the three numbers (Sobel & Lerner; 1995 p.559)

4.2 Analysis

Null hypothesis: There will be no difference between the achievement of the experimental and the control group.

Table 1; shows the percentage marks for the experimental group. The arithmetic mean and standard deviation were calculated.

Table 2; shows the percentage marks for the control group. The arithmetic mean and standard deviation were calculated.

TABLE 1:

X	f	fx	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
4.5	0	0	-54	2916	0
14.5	2	29	-44	1936	3872
24.5	4	98	-34	1156	4624
34.5	5	172.5	-24	576	2880
44.5	6	267	-14	196	1176
54.5	8	436	-4	16	128
64.5	20	1290	6	36	720
74.5	7	521.5	16	256	1792
84.5	6	507	26	676	4056
94.5	2	189	36	1296	2592
	60	3510			21840

$$\sum fx = 3510$$

$$\bar{x} = \frac{3510}{60} = 58.5$$

$$\text{Variance} = \frac{\sum f(x - \bar{x})^2}{\sum f} = 364$$

$$\text{SD} = \sqrt{364} = 19.07$$

TABLE 2:

x	f	fx	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
4.5	2	9	-35.5	1260.25	2520.5
14.5	4	58	-25.5	650.25	2601
24.5	6	147	-15.5	240.25	1441.5
34.5	20	690	-5.5	30.25	605
44.5	14	623	4.5	20.25	283.5
54.5	8	436	14.5	210.25	1682
64.5	4	258	24.5	600.25	2401
74.5	1	74.5	34.5	1190.25	1190.25
84.5	1	84.5	44.5	1980.25	1980.25
94.5	0	0	54.5	2970.25	0
	60	2380			14705.75

$$\bar{x} = \frac{2380}{60} = 40$$

$$\text{Variance} = \frac{14705.75}{60} = 245$$

$$\text{S.D.} = \sqrt{245} = 15.6$$

- a) We take the null hypothesis as $H_o : \mu_e = \mu_c$ where μ_e is the mean for the experimental group and μ_c is the mean for the control group.

The alternative hypothesis says there is a difference.

$$H_1 : \mu_e \neq \mu_c$$

- b) Since the alternative hypothesis is $H_1 : \mu_1 \neq \mu_2$ we use a two-tail test.

c) Since $\alpha = 0.01$

$$\mu_e = 58.5, \sigma_1 = 19.07, n_1 = 60$$

$$\mu_c = 40, \sigma_2 = 15.6, n_2 = 60$$

$$\begin{aligned} \text{Critical values} &= \pm 2.58 \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \pm 2.58 \sqrt{\frac{19.07^2}{60} + \frac{15.6^2}{60}} \\ &= \pm 2.58(3.18) = \pm 8.20 \end{aligned}$$

4.3 Discussion of the result

d) Since $\mu_1 = 58.5$ and $\mu_2 = 40$ then $\mu_1 - \mu_2 = 58.5 - 40 = 18.5$ is in the critical region. Therefore at the 1% level of significance we must reject H_0 and accept H_1 that the mean marks for the two tests are statistically higher.

The experimental group and the control group were very much of the same calibre at the start of the experiment. This results from the fact that the selection procedure of the college applied to each and every one who was admitted at the college. The result is not an outcome of natural growth because both groups had very consistent results in their previous years, that is first and second year for their tests and examination. The same reason regarding similar performance in tests and examination account for the fact that history cannot be the cause of the experimental group's superior achievement.

Class average for the experimental group and the control group for the past three years.

Year	Experimental Group	Control Group
1997	49	52
1998	50	48
1999	54	56

Limitation

Several factors limit the generalisation of the results of this study. These factors are.

1. The students were not representative of pre-service secondary teachers in South Africa. Only students from the previously disadvantaged community were involved.
2. The students were not randomly assigned to the experimental group and the control group. They were in their normal college classes.

Despite these limitations the sample is fairly representative of students of colleges of education of this nature. This results from the fact that these colleges offer the same mathematics curriculum and utilise the same selection procedures.

4.4 Qualitative data

Data on students written work were also collected. In this section we analyse this written work with a view to get a sense of the solution strategies students developed. I had three groups of students in the class namely A, B & C. For the tree plantation (see appendix) I have looked at the work of group A and noted their strategies and tried to analyse my reading on why they did and what they did.

For Group A, I analysed their work as follows:

The group used absolute calculations initially. This means that they concentrated on the given percentages and totals and lost sight of some of the meanings particularly "after one growth year ..."

Their S (small): 40% = 360 trees implies that they obtained 360 at the start of the growth year. This is deduced from the fact that they did the calculation

$\frac{40}{100} \times 600 = 240$ and they indicated the 240 with the word "lost". Then they calculated

$\frac{60}{100} \times 600 = 360$ and they indicated 360 with the word "small" which is assumed to

mean the number of small trees at the start. They proceed in a similar way calculating 20% and 80% of 1200 for the medium trees. However, somehow [there is no written indication] the students must have realised that the absolute calculation strategy was not correct. They then reverted to a totalling and averaging strategy to find the number of trees in each category. Here totals after one year's growth were added and then divided by three. This led to 1 000 trees in each category.

Group B, rewrote the "number of small trees as 600, medium 1200 and larger 1200" forgetting the facts that the numbers that were given in the question was "after the growth year". Secondly they doubted the totals; they took the growth factors as doubling. Possibly cued by the second growth year.

It looks as if group C had the sense of what they were doing. They could comprehend it correctly.

- 1.1 small trees: A
 medium trees: B
 large trees: C

$$\text{After one year } A - \frac{40}{100}A = 600 \Rightarrow A = 1000$$

$$B + \frac{40}{100}A - \frac{20}{1000}B = 1200$$

$$\therefore B + 400 - \frac{1}{2}B = 1200 \Rightarrow B = 1000$$

$$C + \frac{20}{100}B = 1200 \Rightarrow C = 1000$$

1.2 After two years: $A_2 : 600 - \frac{40}{100}(600) = 360$

$$B_2 : 1200 + 360 - \frac{20}{100}(1200) = 1320$$

$$C_2 : 1200 + 1320 = 2520$$

Assumptions: no dead/new trees.

1.3 After fertilizer $A_3 - \frac{40}{100} A_3 = 900$

$$\frac{3}{5} A_3 = 900$$

$$A_3 = 1500 \text{ small}$$

$$B_3 + 1500 - \frac{50}{100} (B_3) = 2500$$

$$\frac{1}{2} B_3 = 1000$$

$$B_3 = 2000 \text{ medium}$$

$$C_3 + 2000 = 5400$$

$$C_3 = 3400 \text{ large}$$

In this chapter I have tried to explain the motivation behind this study. The methodology has been explained. The question paper that I had used to test both the experimental group and control group have also been included. The analysis of the results and the conclusions followed. In the last part I have tried to review the qualitative analysis of the students' work after they had the opportunity of experiencing the problem-centred approach. The next chapter elaborates on my conclusion and recommendations.

Chapter 5

Discussion and conclusion

This study dealt with prospective teachers' development of solution procedures in linear algebra. The major finding of the study is that students who were exposed to a problem-centred approach outperformed students exposed to a traditional approach in a linear algebra test. The students in the experimental group were able to cope with the current innovative thinking on the solution of system of linear equations, and even showed signs of determining meaning, stating purpose and justification. In this chapter we discuss these findings in relation to Curriculum 2005, assessment and support systems. The chapter concludes with recommendations for the use of the problem-centred approach in mathematics teacher education.

5.1 Discussion

5.1.1 Relatedness to Curriculum 2005

Curriculum 2005 is actually a new way of looking at what we want to achieve with our students. It is about developing people who:

- can communicate
- can solve problems
- are confident
- can work with others
- have life skills (Bertram, Botha, Desmond, Dlamini, Johnstone, Ntsingila-khosa, Seery 1997)

Curriculum 2005 is aimed at moving the focus from the traditional approach to an outcome-based teaching approach. The difference between traditional and outcome-based education are listed in the following table.

Traditional	Outcome-based
<ul style="list-style-type: none"> • Passive learners 	<ul style="list-style-type: none"> • Active learners
<ul style="list-style-type: none"> • Rote-learning 	<ul style="list-style-type: none"> • Critical thinking, reasoning reflection
<ul style="list-style-type: none"> • Syllabus is content-based, and broken down into subjects 	<ul style="list-style-type: none"> • An integration of knowledge: learning relevant and connected to real life situation.
<ul style="list-style-type: none"> • Text-book/worksheet bound 	<ul style="list-style-type: none"> • Learner-centred, teacher is facilitator teacher uses group-work and a variety of resources.
<ul style="list-style-type: none"> • Teacher responsible for learning motivation depends on the personality of the teacher. 	<ul style="list-style-type: none"> • Learners take responsibility for their learning. The learners motivated by constant, feedback and affirmation.

If we look at the paradigm shift that has to take place in outcomes-based education, more importance has to be given to students' life experience. Experience is the richest resource for learning; therefore the core methodology of education is the analysis of experience. "New information should always be presented in a context that is familiar to the reader and the context should be established first ... it is reasonably easy to learn something that matches or extends an existing mental model; it is hard to learn something we do not almost already know". (Redish 1994; p.799, cited in Ellis 1995; p.7)

The ideal aim is a process of mutual enquiry that draws ideas out of the students, starting from their own experience. The challenge is firstly, to provide learning experiences that make the understanding real, not something memorized; and then to provide a way of systematizing what has been learnt - drawing out the more general ideas and concepts from it. Secondly, as far as the group process is concerned the learners will be committed to the learning process if they have control or influence over it. This can be a great help in motivation. This means that as far as possible, while remaining true to the chosen educational goal, one should negotiate the content and teaching method of a teaching programme with the learners, giving them some responsibility for its nature and content. In outcome-based education the students take responsibility of their learning. There are individual differences among people and their prior experience; therefore education must make optimal provision for differences in style and pace of learning. This means that the educational programme needs to be designed with a flexibility of learning and teaching method in mind, so as to make optimal provision for differences in style, pace, and context of learning.

There is nothing wrong with studying higher realms of knowledge that have no immediate use, this is a valuable activity and should be encouraged. But the problem arises when uselessness of learning itself becomes the object of pride, with applied research and learning classified as second rate and lesser worth. This is a problem in the majority of developing countries today. This would finally result in the inability of school learners to perform basic tasks of measurement and observation, and use reasoning skills on the job (Lewin 1992; p. 37, cited in Ellis 1995; p.2). In a context such as South Africa today, the urgent need is to make a success of the

reconstruction of the economy of the country. There is a need at this time to provide our population with usable mathematical abilities that can benefit them as individuals and as citizens, and also benefit the economy of the country. The problem-centred approach assist our students to cope with the current innovative thinking

There are two major approaches to the subject material: the first aims to start from general principles to the specific, the second from specific problems, then generalising the underlying principles of wider applicability. If you view the problem-centred approach through a lens you will see the approach has more to do with specific to general. The first approach is a good way to unify and tie up a subject, attaining a deeper understanding of its nature, once its basic ideas and principles are already well understood. Thus the pupils starting with immediate interests, experience, knowledge, abilities and understanding will have a good foundation by learning from specific to general principles.

In my study I have considered aspects of introductory linear algebra from a problem-centred perspective that has much to do with moving from the specific to the general by giving the students the opportunity to experience the context in which the topic is placed. It also addresses the main objectives of Curriculum 2005 which I have discussed in the first part of this chapter.

I am supporting the broad approach provided the teacher does the following effectively.

- Feed back and corrective measures that are regular, diagnostic, and prescriptive;
- use of co-operative learning methods and instrumental enrichment, with compatible assessment method;
- and expectation that almost all the students will eventually succeed in learning the requisite effectively, and so most will at the end attain high marks in the associated lists or other assessment (Ellis 1995; p.14).

In chapter 4, I had indicated the method in which I have conducted my research. It shows very clearly that I have followed the above-mentioned statement and the mean mark of the experimental group is significantly higher from that of the control group. On the basis of my study I have experienced that it was a time consuming exercise.

Problem-centred methods could be applied to the major part of the envisaged mathematics syllabus. Less material will be covered, but it will be understood much better; the resulting increase in confidence and success has potential to defuse much of the prevalent fear of mathematics (Ellis 1995; p.14)

5.1.2 Assessment

Even though the result that I have used to analyse the effect of the problem-centred approach is by a formal test I would like to make some suggestions based on outcome-based education on assessment. The question that I have set as specified in chapter 4 indicates that those problems were from the students' daily life situations. Now let us see what the Department of Education has to say about the

purpose of assessment: the purpose of assessment could include any of the following: diagnosis, evaluation, guidance, grading, selection, prediction and control. (Draft Assessment Policy May 1998).

There is no point in testing students every week if there isn't a good reason for doing it beyond collecting marks for the marks-book. We should be able to say why we are assessing the students, who will benefit from the assessment and why we chose the form of assessment we did. If we can't we should probably think about our assessment practices.

The issue of assessment is a complex one. It is essential that a style of assessment be developed that encourages understanding and problem solving, without discouraging the student. Examinations should not test memory alone, nor be largely based on tricks but rather should test reasoning ability and basic mathematical skills. One of the main problems is the dominance of the examination in our mathematics curriculum. Examinations can certainly play an important role by motivating the student to work. But it is unfair if we ignore the work the student has put into homework and projects during the year. Thus I suggest that a significant part of the assessment should be through class-work and test marks accumulated during the year, and through individual and group projects. Questions set can be phrased to demonstrate understanding and flexibility rather than just ability to complete calculations with an unambiguous answer. For example, instead of requesting that a system of equations to be solved by using the Gaussian elimination method, a question like the tree plantation or even Best Buy can be asked.

5.1.3 Support system

An absolute key to success of a new approach of the kind mentioned in this thesis is an associated system of teacher training and support. The majority of the teachers and teacher training lecturers have limited experience of the kind of approach advocated here, relating the mathematical material to everyday issues, and of the problem-solving approach to learning which is the basis of its teaching. Part of the process of introducing such an approach is to ensure very good in-service training for those teachers who are already teaching. In addition, appropriate approaches to training for new teachers in training colleges and universities are required. They need training in basic methods of educational research and should regard teaching as an ongoing research project into the best educational methods.

5.2 Conclusion

The problem-centred approach aims to show teachers and students the inter-related nature of mathematics and it also makes it clear that mathematics is logically based and a useful subject. For effective implementation of the problem-centred approach what we need is the willingness to consider radical restructuring of the curriculum. I believe the possible improvement in achievement will make this effort worthwhile.

APPENDIX

Some activities based on the problem-centred approach

Best Buys

1. Shopper A pays R20,19 for 2 tins of Nesquik and 1 packet of Pronutro.

Shopper B pays R19,83 for 1 tin of Nesquik and 2 packets of Pronutro.

What is the advertised price of each item?

2. Shopper A pays R16,84 for 3 packets of margarine, 2 packets of Cheese Elites and a Yoghurt.

Shopper B pays R14,15 for 3 yoghurts, a packet of margarine and a packet of Cheese Elites.

Shopper C pays R25,21 for 2 yoghurts, 3 packets of Cheese Elites and 4 packets of Margarine.

What is the advertised price of each item?

Other People's Solutions

The solutions below were given for the BEST BUYS activity:

A			B		
NNN		40.32	N - P		0.36
PPP					
N P		13.34	NNN		20.55
N +		20.19	N		6.85
13.34					
N		6.85	6.85 - P		0.36
P		6.49	P		6.49

C			D		
2	1	20.19	2	1	20.19
1	2	19.83	1	2	19.83
2	1	20.19	2	1	20.19
3	3	40.02	1	-1	0.36
2	1	20.19	2	1	20.19
1	1	13.34	3	0	20.55
1	0	6.85	2	1	20.19
1	1	13.34	1	0	6.85
1	0	6.85	0	1	6.49
0	1	6.49	1	0	6.85

Is (C) more or less the same as (A)?

Are (B) and (D) the same?

	M	C	Y	
A	3	2	1	16.84
B	1	1	3	14.15
C	4	3	2	25.21
	8	6	6	56.2
	4	3	3	28.10

$28.10 - 25.21 = R\ 2.89$ because 4 margarine, 3 cheese and 2 yoghurt costs 25.21

1 margarine, 1 cheese and 1 yoghurt will be $14.15 - 2 \times 2.89 = 8.37$

3 margarine, 3 cheese and 3 yoghurt will be $3 \times 8.37 = 25.11$

But 4 margarine, 3 cheese and 3 yoghurt is 28.10 so 1 margarine is

$$28.10 - 25.11 = 2.99$$

1 cheese and 1 yoghurt will be $8.37 - 2.89 = 5.38$ and 1 cheese will be

$$5.38 - 2.89 = 2.49$$

Rewrite this student's method in the form of (C) or (D).

The methods (C) and (D) will be called the row elimination method.

Find the answers of TREE PLANTATION by using the row elimination method.

Tree Plantation

1. In a particular tree plantation there is three categories of trees: Small, Medium and Large. It was found that over a growth year 40% of the small trees become medium ones and 20% of the medium ones become large trees. After one growth year there are 600 small trees; 1200 medium trees and 1200 large trees.
 - 1.1 How many trees were there in each category at the beginning of the growth year?
 - 1.2 How many trees are there in each category at the end of the second growth year? What assumptions are you making?

2. The plantation management decides to use fertiliser to accelerate the growth of trees. This causes the growth over a year to be as follows: 40% of the small trees become medium trees and 50% of the medium trees become large ones.

After a year's growth with the fertiliser treatment there are 900 small trees; 2500 medium ones and 5400 large trees.

How many trees were there at the start of this growth year?

JUST LETTERS

Solve the following systems of equations using the row elimination method.

1. $a + 2b + 3c = 9$
 $4a + 5b + 6c = 24$
 $3a + b - 2c = 4$

2. $a - 2b + 3c = 0$
 $4a + b - c = 0$
 $2a - b + 3c = 0$

3. $2a + 3b - c = 9$
 $-2a + b + 3c = 7$
 $-6a - b + 12c = 15$

(Julie 1996)

REFERENCES

- Bertram, C, Botha, W. & Desmond, S. (Eds.) (1997): **Curriculum 2005**: Pretoria:
The media in education trust of the National Department of Education.
- Brase, H. & Brase, P. (1983): **Understanding Statistics**: D.C. Heath and
Company.
- Confrey, J. (1996): **Towards a framework for a Constructivist Instruction**;
New York, Cornell University.
- Davies, R.B. (1984): **Learning Mathematics; The Cognitive Science Approach to
Mathematics Education**: London. Croom Helm.
- Dienes, Z. P. (1964): **The Power of Mathematics**: London, Hutchinson.
- Ellis, G.F.R. (1995): **The Syllabus for School Mathematics**: University of Cape
Town: Department of Applied Mathematics.
- Gauteng Department of Education (1996): Syllabus for Std 8, 9, 10.
- Halmos, P. (1995): **Linear Algebra Problem Book**: Washington, The American
Mathematical Association.
- Human, P. (1996): **Class notes**: Department of Didactics, University of Stellenbosch.

Julie, C. (1996): **Activities Based on the Problem-Centred Approach:**
Department of Didactics, University of the Western Cape.

Katz, V.J. (1995): Historical Ideas in Teaching Linear Algebra. In Swetz, F.
Faurel, J. Bekken, O. Johansson, B. (1995): **Learn from the Masters:**
The Mathematical Association of America.

Laridon, P., Brink, M.E., Burgess, A.G., Jawurek, A.M., Kitto, A.L., Myburgh, M.J.,
Pike, M.R., Rhodes-Houghton, R.H., van Rooyen, R.P., (1985): **Classroom**
Mathematics of Std 8, 9, 10: Johannesburg: Lexicon.

Lehman & Richard, S. (1991): **Statistics and Research Design in the**
Behavioural Sciences: California: Wadsworth Publishing Company.

Moodley, M., Njisani, R.A. & Presmeg, N. (Eds) (1992): **Mathematics**
Education for In-service teachers and Pre-service: Pietermaritzburg:
Shuter & Shooter.

Murray, H. Olivier, A. & Human, P. (1993): Voluntary Interaction Groups for
Problem-centred Learning. **Proceedings of the Seventeenth**
International Conference for the Psychology of Mathematics
Education, Vol II, 73-80. Tsukuba, Japan.

- Olivier, A. (1989): Handling pupil's misconceptions: In Moodley, M., Njisani, R.A. & Presmeg, N. (Eds) **Mathematics Education for In-Service teachers and Pre-service** (pp. 193-209). Pietermaritzburg: Shuter & Shooter.
- Olivier, A. (1984): **Introductory Algebra**; In-service education course Cape: Education Department.
- Sobel, M. & Lerner, N. (1995): **Precalculus Mathematic**: Eagelwood Cliffs, A Simon & Schuster Company.
- Strang, G. (1980): **Linear Algebra and Applications**: Orlando, Florida: Academic Press, Inc.
- Sfard, A. & Linchevski, L. (1994): The gains and the pitfalls of reification - the case of algebra: **Educational Studies in Mathematics**, 26, 191-228. Kluwer Academic Publishers.
- Third International Mathematics and Science Study (1996). TIMSS International Study Centre. Boston College. [Online document] [http://timss.bc.edu/timss1995i/TIMSPDF/C Hilite.pdf](http://timss.bc.edu/timss1995i/TIMSPDF/C%20Hilite.pdf)**
- Usiskin, Z. (1998): *Conceptions of School Algebra and uses of Variables*. In A.F. Coxford (Eds) **The ideas of Algebra, K-12**, 8-19. Reston, Virginia. The National Council of Teachers of Mathematics.

Wankang & Kilpatrick ,J. (1992): **Didactic Transportation in Mathematics**

Textbooks: White Rock, British Columbia, FLM Publishing Association.